THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 2

§11) 1) (b) Note that $1-\sqrt{3}i = 2e^{-i(\frac{\pi}{3})}$. Therefore,

$$(1-\sqrt{3}i)^{1/2} = \{\sqrt{2}e^{-i(\frac{\pi}{6})}, \sqrt{2}e^{-i(\frac{\pi}{6})+i\pi}\} = \left\{\pm\frac{\sqrt{3}-i}{\sqrt{2}}\right\}.$$

§11) 4) (b) Note that $8 = 8e^{i(0)}$. Therefore,

$$\begin{aligned} 8^{1/6} &= \left\{\sqrt{2}, \sqrt{2}e^{i(\frac{\pi}{3})}, \sqrt{2}e^{i(\frac{2\pi}{3})}, \sqrt{2}e^{i(\frac{3\pi}{3})}, \sqrt{2}e^{i(\frac{4\pi}{3})}, \sqrt{2}e^{i(\frac{5\pi}{3})}\right\} \\ &= \left\{\pm\sqrt{2}, \pm\frac{1+\sqrt{3}i}{\sqrt{2}}, \pm\frac{1-\sqrt{3}i}{\sqrt{2}}\right\}. \end{aligned}$$

The principal root is given by $\sqrt{2}$.

§11) 8) (a)

$$\begin{aligned} az^2+bz+c &= 0 \\ z^2+\frac{b}{a}z+\frac{c}{a} &= 0 \\ z^2+2\left(\frac{b}{2a}\right)z+\left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2-\frac{c}{a} \\ (z+\frac{b}{2a})^2 &= \frac{b^2-4ac}{4a^2} \\ z+\frac{b}{2a} &= \sqrt{\frac{b^2-4ac}{4a^2}} \\ z &= \frac{-b+\sqrt{b^2-4ac}}{2a}. \end{aligned}$$

(b) Consider the equation $z^2+2z+(1-i)=0$. By quadratic formula, we have

$$\begin{aligned} z &= \frac{-2+\sqrt{2^2-4(1)(1-i)}}{2} \\ &= -1+\sqrt{i} \\ &= -1+\sqrt{e^{i(\frac{\pi}{2})}} \\ &= \left\{-1+e^{i(\frac{\pi}{4})}, -1+e^{i(\frac{\pi}{4}+\pi)}\right\} \\ &= \left\{-1\pm\left(\frac{1+i}{\sqrt{2}}\right)\right\} \\ &= \left\{\left(-1+\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}i,\ \left(-1-\frac{1}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}i\right\}. \end{aligned}$$

§14) 1) (a) $f(z) = \dfrac{1}{z^2+1}$ is not well-defined $\iff z^2+1=0 \iff z=\pm i$.

Hence the domain of definition of $f(z) = \dfrac{1}{z^2+1}$ is given by $\mathbb{C}\backslash\{\pm i\}$.

(b) Since $\frac{1}{z} \neq 0$, $f(z) = \operatorname{Arg}\left(\frac{1}{z}\right)$ is not well-defined $\iff$ $\frac{1}{z}$ is not well-defined $\iff$ $z = 0$.

Hence the domain of definition of $f(z) = \frac{1}{z^2+1}$ is given by $\mathbb{C}\backslash\{0\}$.

§14) 2) (a)

$$\begin{aligned} f(z) &= z^3 + z + 1 \\ &= (x+iy)^3 + (x+iy) + 1 \\ &= x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y). \end{aligned}$$

(b)

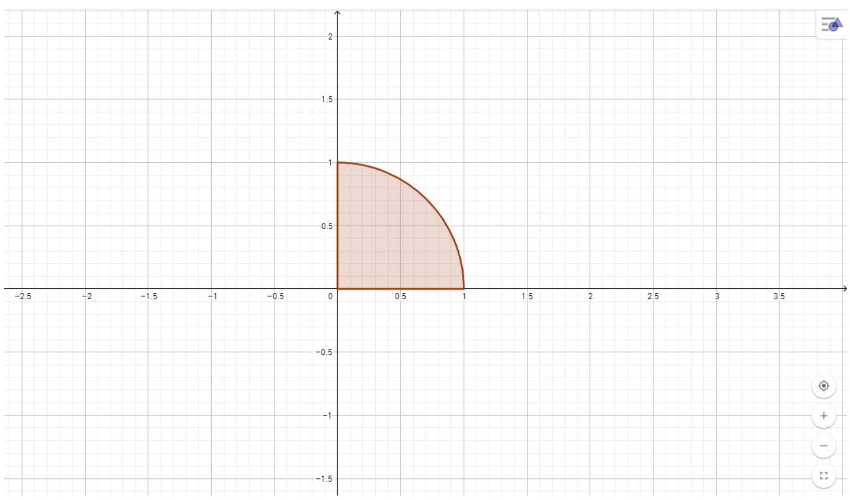
$$\begin{aligned} f(z) &= \frac{\overline{z}^2}{z} \\ &= \frac{\overline{z}^3}{|z|^2} \\ &= \frac{(x-iy)^3}{|x+iy|^2} \\ &= \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{x^2+y^2} \\ &= \frac{x^3 - 3xy^2}{x^2+y^2} + i\frac{y^3 - 3x^2y}{x^2+y^2}. \end{aligned}$$

§14) 4)

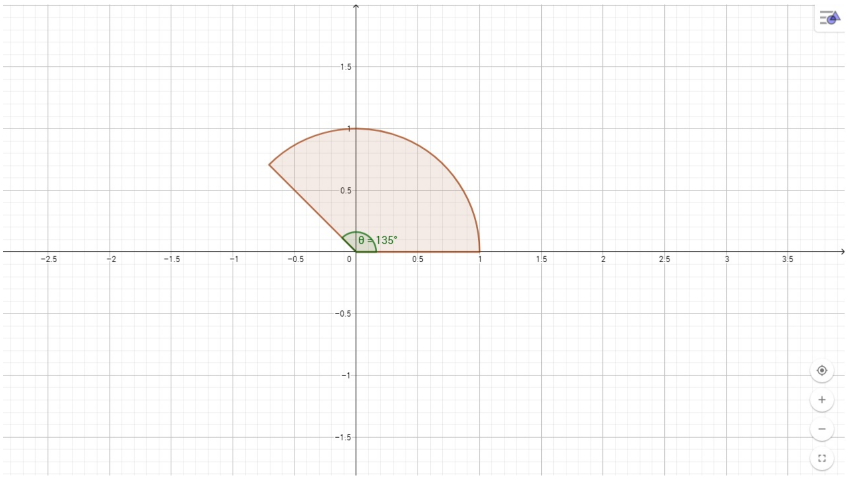
$$\begin{aligned} f(z) &= z + \frac{1}{z} \\ &= re^{i\theta} + r^{-1}e^{-i\theta} \\ &= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta. \end{aligned}$$

§14) 8) The following pictures are drawn by Geogebra (https://www.geogebra.org/classic).

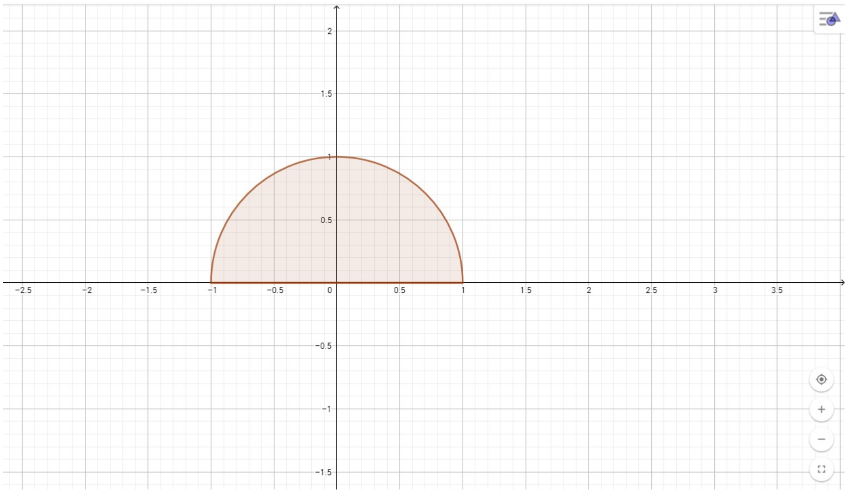
(a)



(b)



(c)



§18) 3) (a) $\displaystyle\lim_{z\to z_0}\frac{1}{z^n}=\frac{1}{\displaystyle\lim_{z\to z_0} z^n}=\frac{1}{z_0^n}.$

(b) $\displaystyle\lim_{z\to i}\frac{iz^3-1}{z+i}=\frac{\displaystyle\lim_{z\to i}(iz^3-1)}{\displaystyle\lim_{z\to i}(z+i)}=\frac{i(i)^3-1}{i+i}=0.$

(c) $\displaystyle\lim_{z\to z_0}\frac{P(z)}{Q(z)}=\frac{\displaystyle\lim_{z\to z_0} P(z)}{\displaystyle\lim_{z\to z_0} Q(z)}=\frac{P(z_0)}{Q(z_0)}.$

§18) 10) (a) Since $\displaystyle\lim_{z\to 0}\frac{4\left(\frac{1}{z}\right)^2}{\left(\frac{1}{z}-1\right)^2}=\lim_{z\to 0}\frac{4}{(1-z)^2}=4$, we have $\displaystyle\lim_{z\to\infty}\frac{4z^2}{(z-1)^2}=4.$

(b) Since $\displaystyle\lim_{z\to 1}(z-1)^3=0$, we have $\displaystyle\lim_{z\to 1}\frac{1}{(z-1)^3}=\infty.$

(c) Since $\displaystyle\lim_{z\to 0}\frac{\left(\frac{1}{z}\right)-1}{\left(\frac{1}{z}\right)^2+1}=\lim_{z\to 0}\frac{z-z^2}{1+z^2}=0$, we have $\displaystyle\lim_{z\to\infty}\frac{z^2+1}{z-1}=\infty.$

§18) 11) (a) If $c = 0$, since $ad - bc \neq 0$, a and d are non-zero complex numbers. Also,

$$\lim_{z\to 0} \frac{1}{T(1/z)} = \lim_{z\to 0} \frac{d}{\frac{a}{z} + b} = \lim_{z\to 0} \frac{dz}{a + bz} = 0.$$

Hence $\lim\limits_{z\to\infty} T(z) = \infty$.

(b) Since

$$\lim_{z\to 0} T(1/z) = \lim_{z\to 0} \frac{\frac{a}{z} + b}{\frac{c}{z} + d} = \lim_{z\to 0} \frac{a + bz}{c + dz} = \frac{a}{c},$$

we have $\lim\limits_{z\to\infty} T(z) = \dfrac{a}{c}$.

Also, since $ad - bc \neq 0$, we also have $\dfrac{ad}{c} - b \neq 0$. Since

$$\lim_{z\to -\frac{d}{c}} \frac{1}{T(z)} = \lim_{z\to -\frac{d}{c}} \frac{cz + d}{az + b} = \frac{0}{-\frac{ad}{c} + b} = 0,$$

we have $\lim\limits_{z\to -\frac{d}{c}} T(z) = \infty$.